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Quantum circuit implementation of the optimal information–disturbance tradeoff of maximally entangled states

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Abstract

We give a direct derivation for the information–disturbance tradeoff in estimating a maximally entangled state, which was first obtained by Sacchi (2006 *Phys. Rev. Lett.* **96** 220502) in terms of the covariant positive operator valued measurement (POVM) and Jamiołkowski's isomorphism. We find that, the Cauchy–Schwarz inequality, which is one of the most powerful tools in deriving the tradeoff for a single-particle pure state still plays a key role in the case of the maximal entanglement estimation. Our result shows that the inequality becomes equality when the optimal tradeoff is achieved. Moreover, we demonstrate that such a tradeoff is physically achievable with a quantum circuit that only involves single- and two-particle logic gates and single-particle measurements.

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1. Introduction

Quantum measurement is one of the most important issues in quantum information processing, such as quantum key distribution, quantum teleportation and, especially, quantum computation. However, quantum mechanics further imposes some limitations on quantum information extraction from the unknown quantum state. In particular, there is not a quantum measurement on the quantum system without introducing any disturbance. The more information we gain, the more its state has to be disturbed. Actually, there exists a precise tradeoff between the amount of information we gain and the disturbance caused on the quantum system. Since it was first proposed in [1, 2], the tradeoff between information gain and quantum state disturbance has received a wide and extensive attention [3–19] .

In [3], Banaszek reported an optimal tradeoff between information gain and state disturbance for a completely unknown finite-dimensional single-particle pure state. In this study, the tradeoff in the scenario of quantum state estimation is exactly obtained. Following this seminal work, a lot of work has been done in deriving the optimal tradeoff for different sets of unknown quantum states. This includes the optimal quantum estimation in the following cases: (1) a partially known finite dimensional pure state on circles [4]; (2) many copies of identically prepared pure qubits [5, 6]; (3) completely unknown maximally entangled bipartite pure state [7]; (4) Gaussian [8] or Non-Gaussian [9] continuous-variable systems; and (5) spin coherent states [10] and recently a tradeoff in the quantum discrimination of non-orthogonal pure state has also been reported [11, 12]. Moreover, explicit physical schemes to achieve the optimal tradeoff have already been proposed. In [13, 14], it was shown that the optimal tradeoff for completely unknown single-particle pure state can be obtained by a quantum circuit that consists of single- and two-qudit logic gates and individual measurements. In laboratory, experiments achieving the optimal tradeoff for photon polarization qubit [15] and for coherent state quantum system [16] have also been reported.

Let us now give a general formalism of the problem. Assuming a quantum state $\rho = |\psi\rangle\langle\psi|$ is homogeneously picked (or according to an assigned *a priori* probabilistic distribution) from a given set Ω , one performs a quantum measurement on the unknown state and then infers what state will be based on his measurement outcome. Consequently, the priori quantum state is inevitably distorted. Both the information one gains and the amount of the disturbance can be quantified with fidelities. In precise, let us consider the most general quantum measurement, i.e., positive operator valued measurement (POVM) [20, 21]. Such a measurement scheme is always described by a set of positive operators $\left\{\prod_r\right\}$ acting on the unknown quantum state ρ. The measurement is probabilistic: the probability of the result r is $p_r = \mathrm{Tr}\left[\prod_r \rho\right]$. When the measurement outcome r is observed, the general form for the post-measurement quantum state can be written in the form of a series of operators $\{A_{r\mu}\}$:

$$\rho_r' = \sum_{\mu} A_{r\mu}\rho A_{r\mu}^{\dagger}\big/ p_r, \tag{1}$$

where the operators A_μ are named as *Kraus operators* and follow the relation $\prod_r = \sum_\mu {A_{r\mu}}^{\dagger} A_{k\mu}$. The trace-preserving condition of the measurement can be given by $\sum_r \prod_r = I$. Conditional on the measurement outcome, one can establish some inference rule being $r \to \rho_r = |\phi_r\rangle\langle\phi_r|$ as an estimation of the unknown state ρ. Thus, the fidelity (or the overlap) between inferred state $|\phi_r\rangle$ and ρ is a good characterization of the information one gain from the measurement. By averaging over all the possible outcomes, the information gain, for a given input ρ, can be written as

$$\mathcal{G}_\rho = \sum_{r\mu} \mathrm{Tr}\left[A_{r\mu}^{\dagger} A_{r\mu}\rho\right]\mathrm{Tr}[\rho\rho_r]. \tag{2}$$

Similarly, the amount of disturbance caused by the measurement can be quantified by evaluating the overlap between the input state ρ and the output state $\rho' = \sum_r p_r\rho_r' = \sum_{r\mu} A_{r\mu}\rho A_{r\mu}^{\dagger}$:

$$\mathcal{F}_\rho = \mathrm{Tr}\left[\rho'\rho\right] = \sum_{r\mu} \mathrm{Tr}\left[A_{r\mu}\rho A_{r\mu}^{\dagger}\rho\right]. \tag{3}$$

The fidelity $\mathcal{F}_\rho$ and $\mathcal{G}_\rho$ are all dependent on the special choice of the input state ρ. In total, the relevant quantities that evaluate the measurement procedure with respect to the whole state set Ω can be given by further averaging over all the possible states, or namely, over the set Ω:

$$\mathcal{F} = \int_\Omega \mathrm{d}\rho \sum_{r\mu} \mathrm{Tr}\left[A_{r\mu}\rho A_{r\mu}^{\dagger}\rho\right], \tag{4}$$

$$\mathcal{G} = \int_{\Omega} \mathrm{d}\rho \sum_{r\mu} \mathrm{Tr}\left[A_{r\mu}^{\dagger} A_{r\mu} \rho\right] \mathrm{Tr}[\rho\rho_r]. \tag{5}$$

In the literature, these two quantities, $\mathcal{F}$ and $\mathcal{G}$, are always referred to as *output fidelity* and *estimation fidelity*, respectively' [3]. Quantum mechanics imposes some constraints on the relation between $\mathcal{F}$ and $\mathcal{G}$. For a given value of $\mathcal{F}$, there exists an upper bound value for $\mathcal{G}$ and no physically reliable measurement can be found to beat such an upper bound. Intuitively, with the $\mathcal{F}$ increasing from its minimum to maximum, the physically achievable supermium value of $\mathcal{G}$ will violate from its maximum to minimum. The exact tradeoff between $\mathcal{F}$ and $\mathcal{G}$ is naturally imposed by quantum mechanics and is what we are mainly concerned.

In [7], based on Jamiołkowski representation [22] and Schur's lemma [23] for unitary group representation, Sacchi exploited the covariant quantum operators and obtained the optimal information–disturbance tradeoff in estimating an unknown maximally entangled state. In this paper, we use an alternative approach and regain the optimal tradeoff bound for the estimation of the completely unknown maximally entangled state. Our result will be obtained, in a similar way as Banaszek utilized in [3], by the direct derivation of quantum fidelities and by means of the Cauchy–Schwarz inequality. It turns out that the Cauchy–Schwarz inequality will become an equality when the optimal tradeoff is attained. We then suggest a quantum circuit implementation for the estimation of unknown entangled states. Our schemes are *optimal* in the sense that corresponding fidelities saturate the bound of the tradeoff.

This paper is organized as follows. In section 2, we will adopt Banaszek's method [3] and give an re-examination of the tradeoff for the maximally entangled state. In section 3, we describe the quantum logic circuit realization which only involves single- and two-particle logic gates and single-particle measurements. The simplest example for the $D = 2$ (qubits) case is described in detail, whereas its generalization to an arbitrary D-dimensional case is given briefly. Finally, a conclusion follows in section 4.

2. Rederivation of the tradeoff for the maximally entangled state

Let us now first provide some notations that will be frequently utilized in the rest of the paper. When considering the bipartite pure state, it is convenient to exploit the notation of the Liouville space [24, 25] . According to the Liouville formalism, a bipartite state vector in the Hilbert space $\mathcal{H}_D \otimes \mathcal{H}_D$ can be identified with a single-particle operator acting on $\mathcal{H}_D$. In particular, for the maximal entanglement pure state, the identification is much more simple and powerful. Each maximal entangled state $|\psi_g\rangle$ in $\mathcal{H}_D \otimes \mathcal{H}_D$ will then be uniquely written as

$$|\psi_g\rangle = \frac{1}{\sqrt{D}} |U_g\rangle\!\rangle = \frac{1}{\sqrt{D}} U_g \otimes I |I\rangle\!\rangle, \tag{6}$$

with U_g denoting a $D \times D$ unitary matrix and double ket representation defined by $|A\rangle\!\rangle \equiv \sum_{ij} \langle i|A|j\rangle |i\rangle|j\rangle$. Here g is an element in the Lie group $SU(D)$ and furthermore, when g runs through the group $SU(D)$, we run through the whole set of different maximal entanglement states Ω. This is quite a useful conclusion and one will see that the integration in equation (5) directly boils down to a group average over $SU(D)$. With all these notations, we can rewrite the fidelities in equation (5) with

$$\mathcal{F} = \frac{1}{D^2} \int_{SU(D)} \mathrm{d}g \sum_{r\mu} |\langle\!\langle U_g|A_{r\mu}|U_g\rangle\!\rangle|^2, \tag{7}$$

$$\mathcal{G} = \frac{1}{D^2}\int_{SU(D)} \mathrm{d}g \sum_{r\mu} \langle\langle U_g|A^\dagger_{r\mu}A_{r\mu}|U_g\rangle\rangle|\langle\phi_r|U_g\rangle\rangle|^2, \tag{8}$$

in which we have considered the normalized invariant *Harr* measurement dg over the group , namely, $\int_{SU(D)} \mathrm{d}g = 1$ [23].

Before further precessing, there is one important assumption that should be justified here. Here and after, we will investigate the projective measurement which involves only a single value of μ in the sum equation (1), namely, the POVM $\{A_r\}$. Such an assumption will not introduce the loss of any generality [18]. Actually, to justify this, one can use the polar decomposition of an arbitrary operator $A_{r\mu} = U_{r\mu}P_{r\mu}$, with U being unitary and P being positive. If $P_{r\mu}$ does not vary with μ, then the value of μ does not contain any additional information about the initial state and can be neglected. If $P_{r\mu}$ does vary with μ, then the value of μ represents the information that is not gathered by the POVM $\{A_r\}$, but can be gathered with a new POVM $\{A_{r\mu}\}$ where both r and μ become measurement results and where the upper bound for $\mathcal{F}$ and $\mathcal{G}$ can still be achieved. Furthermore, in terms of output fidelity, the single term Kraus operator (see equation (1)) always give a minimally disturbing way of the POVM measurement (see theorem 5 in [18]) . In what follows, we will omit the second index μ and consider the POVM: $\{A_r\}$ with trace preserving condition $\sum_r {A_r}^\dagger A_r = I$.

We will now start by deriving the fidelities $\mathcal{F}$ and $\mathcal{G}$. One can introduce the identity operator $\sum_{mn}|mn\rangle\langle mn| = I\otimes I$ and insert it into equation (7):

$$\begin{aligned}\mathcal{F} &= \frac{1}{D^2}\int \mathrm{d}g \sum_r\sum_{mn}\sum_{pq}\langle\langle U_g|mn\rangle\langle mn|A_r|U_g\rangle\rangle\langle\langle U_g|A^\dagger_r|pq\rangle\langle pq|U_g\rangle\rangle \\ &= \frac{1}{D^2}\sum_{mn}\sum_{pq}\sum_r\langle mn|A_r\mathcal{M}_{mnpq}A^\dagger_r|pq\rangle.\end{aligned} \tag{9}$$

In equation (9), we define a $D^2\times D^2$ matrix $\mathcal{M}_{mn,pq}$:

$$\begin{aligned}\mathcal{M}_{mnpq} &\equiv \int \mathrm{d}g\langle\langle U_g|mn\rangle\langle pq|U_g\rangle\rangle|U_g\rangle\rangle\langle\langle U_g| \\ &= \int \mathrm{d}g\mathrm{Tr}_{12}[(|mn\rangle\langle pq|\otimes I\otimes I)(|U_g\rangle\rangle\langle\langle U_g|\otimes|U_g\rangle\rangle\langle\langle U_g|)],\end{aligned}$$

where in the second line we have extended the original Hilbert space and the subscript '12' indicates that the partial trace is performed with respect to the first two Hilbert spaces. The explicit form of $\mathcal{M}_{mnpq}$ can be further simplified with the help of Schur's lemma for the unitary group representation [23]:

$$\int \mathrm{d}g|U_g\rangle\rangle\langle\langle U_g|\otimes|U_g\rangle\rangle\langle\langle U_g| = \frac{2}{D(D-1)}P_A^{(1,3)}\otimes P_A^{(2,4)} + \frac{2}{D(D+1)}P_S^{(1,3)}\otimes P_S^{(2,4)},$$

where $P_A^{(i,j)}$ is the projector on the $\frac{D(D-1)}{2}$-dimensional asymmetry subspace of the Hilbert space $\mathcal{H}_D^{(i)}\otimes\mathcal{H}_D^{(j)}$ and $P_S^{(i,j)} = I\otimes I - P_A^{(i,j)}$ is the $\frac{D(D+1)}{2}$-dimensional symmetry subspace. Note that the indices 1, 2, 3 and 4 here indicate the relevant Hilbert space which should be distinguished from the qub(d)its of the quantum circuit implementation in section 3.

Then, with some algebra, we obtain that

$$\mathcal{M}_{mnpq} = \frac{1}{D^2-1}\left[\delta_{mp}\delta_{nq}I\otimes I + |mn\rangle\langle pq| - \frac{1}{D}(\delta_{mp}I\otimes|n\rangle\langle q| + \delta_{nq}|m\rangle\langle p|\otimes I)\right]. \tag{10}$$

Putting equations (9) and (10) together, one finds

$$\mathcal{F}=\frac{1}{D^2-1}\left[1+\frac{1}{D^2}\sum_r |\mathrm{Tr}\,[A_r]|^2\right]$$

$$-\frac{1}{(D^2-1)D^3}\left[\sum_{r,p,i}\left(\left|\sum_q \langle \mathrm{i}q|A_r|pq\rangle\right|^2+\left|\sum_q \langle q\mathrm{i}|A_r|qp\rangle\right|^2\right)\right] \tag{11}$$

$$\leqslant \frac{1}{D^2-1}+\frac{D^2-2}{D^4(D^2-1)}\sum_r |\mathrm{Tr}\,[A_r]|^2. \tag{12}$$

Now let us turn our attention to the estimation fidelity fidelity $\mathcal{G}$. According to equation (6), we can also rewrite the maximally entangled state $|\phi_r\rangle$ that we infer as

$$|\phi_r\rangle=\frac{1}{\sqrt{D}}|W_r\rangle\rangle=\frac{1}{\sqrt{D}}W_r\otimes I|I\rangle\rangle. \tag{13}$$

Substituting equation (13) into equation (8) and changing the integration measure according to $|U_g\rangle\rangle\rightarrow W_r\otimes I|U_g\rangle\rangle$, we will see

$$\mathcal{G}=\frac{1}{D^3}\int_{SU(D)}\mathrm{d}g\sum_r \langle\langle U_g|A_r^\dagger A_r|U_g\rangle\rangle|\langle\langle I|W_r^\dagger\otimes I|U_g\rangle\rangle|^2$$

$$=\frac{1}{D^3}\int \mathrm{d}g\sum_r \langle\langle U_g|\left(W_r^\dagger\otimes I\right)A_r^\dagger A_r(W_r\otimes I)|U_g\rangle\rangle\langle\langle I|U_g\rangle\rangle\langle\langle U_g|I\rangle\rangle$$

$$=\frac{1}{D^3}\sum_r \mathrm{Tr}\big[\big(\big(W_r^\dagger\otimes I\big)A_r^\dagger A_r(W_r\otimes I)\big)\mathcal{M}_\Pi\big], \tag{14}$$

in which we have defined

$$\mathcal{M}_\Pi\equiv\int \mathrm{d}g|U_g\rangle\rangle\langle\langle U_g|\langle\langle I|U_g\rangle\rangle\langle\langle U_g|I\rangle\rangle \tag{15}$$

$$=\int \mathrm{d}g\mathrm{Tr}_{12}[(|I\rangle\rangle\langle\langle I|\otimes I\otimes I)(|U_g\rangle\rangle\langle\langle U_g|\otimes|U_g\rangle\rangle\langle\langle U_g|)]. \tag{16}$$

By Schur's lemma, the integral in $\mathcal{M}_\Pi$ can be easily evaluated, which yields

$$\mathcal{M}_\Pi=\frac{1}{D^2-1}\left[\left(D-\frac{2}{D}\right)I+|I\rangle\rangle\langle\langle I|\right]. \tag{17}$$

Hence,

$$\mathcal{G}=\frac{1}{D^2(D^2-1)}\left[(D^2-2)+\sum_r \mathrm{Tr}\left[\left(\left(W_r^\dagger\otimes I\right)A_r^\dagger A_r(W_r\otimes I)\right)|I\rangle\rangle\langle\langle I|\right]\right]$$

$$=\frac{1}{D^2(D^2-1)}\left[(D^2-2)+\sum_r \langle\phi_r|A_r^+A_r|\phi_r\rangle\right]. \tag{18}$$

In equations (12) and (18), we have already presented the expression of $\mathcal{F}$ and $\mathcal{G}$. They are both functions of the concrete measurement operator A_r. In fact, only the eigenvalue and the corresponding eigenstate are concerned with the optimal tradeoff. In order to see this, we can resort to the singular value decomposition (SVD) [26] of operators A_r. According to the SVD theory, any complex matrix A_r can be rewritten as

$$A_r=V_r\Lambda_r T_r$$

where V_r and T_r are both unitary matrices, and Λ_r is a semipositive definite diagonal matrix

$$\Lambda_r = \mathrm{diag}\{\lambda_0^r, \lambda_1^r, \lambda_2^r \ldots \lambda_{D^2-1}^r\} \tag{19}$$

with its diagonal elements arranged in the decreasing order $\lambda_0^r \geqslant \lambda_1^r \geqslant \lambda_2^r \geqslant \ldots \lambda_{D^2-1}^r \geqslant 0$. In general, the elements λ_i^r are not the eigenvalues of A_r. However, since what we are concerned is the optimal tradeoff, there will be no loss of generality if we considered the Hermitian measurement operators. This is because, in equation (18)

$$\langle\phi_r|A_r^\dagger A_r|\phi_r\rangle = \langle\phi_r|T_r^\dagger \Lambda_r^\dagger \Lambda T_r|\phi_r\rangle \leqslant (\lambda_0^r)^2 \tag{20}$$

and in equation (12)

$$|\mathrm{Tr}\,[A_r]| = \left|\sum_{i=0}^{D^2-1}\langle i|T_r V_r \Lambda_r|i\rangle\right| \leqslant \sum_{i=0}^{D^2-1}\lambda_i^r|\langle i|T_r V_r|i\rangle| \leqslant \sum_{i=0}^{D^2-1}\lambda_i^r. \tag{21}$$

Note that the last inequality sign in equation (21) is reached when $T_r = V_r^\dagger$, which guarantees that the operator A_r is Hermitian. From equations (20) and (21), it is clear that $\mathcal{F}$ is non-decreasing and the optimality is preserved if we continue our proof with Hermitian measure operators.

To complete our rederivation, it is convenient to introduce the vector representation that has already been used in deriving the tradeoff for the single-particle pure state [3]. Define D^2 real vectors $\mathbf{v}_i = (\lambda_i^0, \lambda_i^1, \lambda_i^2 \ldots \lambda_i^{D^2-1})$ ($i = 0, 1, \cdots, D^2 - 1$). The term of summation over outcome $\{r\}$ in equations (12) and (18) can now be evaluated as

$$f = \sum_r |\mathrm{Tr}\,[A_r]|^2 \leqslant \sum_r \left(\sum_i \lambda_i^r\right)^2 = \sum_{i,j=0}^{D^2-1}\mathbf{v}_i\mathbf{v}_j, \tag{22}$$

$$g = \sum_r \langle\phi_r|A_r^+ A_r|\phi_r\rangle = \sum_r (\lambda_0^r)^2 = |\mathbf{v}_0|^2. \tag{23}$$

By the Cauchy–Schwarz inequality,

$$f \leqslant \sum_{i,j=0}^{D^2-1}|\mathbf{v_i}||\mathbf{v_j}| = \left(\sum_{i=0}^{D^2-1}|\mathbf{v_i}|\right)^2 = \left(\sqrt{g} + \sum_{i=1}^{D^2-1}|\mathbf{v_i}|\right)^2 \tag{24}$$

and moreover

$$\sum_{i=1}^{D^2-1}|\mathbf{v_i}| \leqslant \sqrt{(D^2-1)\sum_{i=1}^{D^2-1}|\mathbf{v_i}|^2} = \sqrt{(D^2-1)(D^2-g)}. \tag{25}$$

We then have

$$\sqrt{f} \leqslant \sqrt{g} + \sqrt{(D^2-1)(D^2-g)}. \tag{26}$$

Finally, by substituting equations (22) and (23) into equations (12) and (18), we have

$$\mathcal{F} = \frac{1}{D^2-1} + \frac{D^2-2}{D^4(D^2-1)}f,$$
$$\mathcal{G} = \frac{1}{D^2(D^2-1)}[(D^2-2)+g], \tag{27}$$

and optimal tradeoff between operation fidelity and estimation fidelity can be retrieved from equation (26):

$$\sqrt{\left(\mathcal{F} - \frac{1}{D^2-1}\right)\frac{D^2}{D^2-2}} \leqslant \sqrt{\mathcal{G} - \frac{D^2-2}{D^2(D^2-1)}} + \sqrt{(D^2-1)\left(\frac{2}{D^2} - \mathcal{G}\right)}. \tag{28}$$

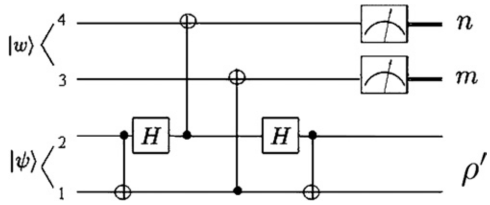

Figure 1. The quantum circuit constructed for achieving the optimal tradeoff for maximally entangled state ($D = 2$).

Up to now, we have re-examined the optimal tradeoff for themaximal entangled state which has already been obtained by Sacchi in [7]. The equality sign holds when the D^2 vectors $\mathbf{v_0}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{D^2-1}}$ are colinear. In the following, we will show that the optimal tradeoff is physical implementable by constructing a concrete quantum circuit.

3. Quantum circuit in estimating maximally entangled states

In this section, we propose a quantum circuit for realizing the optimal estimation of the entangled state. First, we give a description of the simplest case of $D = 2$ in detail. Then, we show that the scheme can be easily generalized to entangled qudits and we describe the corresponding circuit briefly.

The scheme, shown in figure 1, consists of four CNOT gates, two Hadamard gates and two single-qubit measurements. Qubits 1 and 2 are the unknown maximal entangled states that are to be measured. To implement the tradeoff, we further introduce two auxiliary qubits, qubits 3 and 4 , which are initially prepared in a partial entanglement state

$$|w\rangle_{34} = \cos\theta|00\rangle + \gamma\sin\theta|{+}{+}\rangle, \gamma = \frac{\sqrt{1+4\tan^2\theta}-1}{2\tan\theta},$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\theta\left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right)$ is a control parameter which determines the entanglement degree of the ancillary qubit. We will see that this is also the very parameter that determines the information transfer between the estimation fidelity and output fidelity. The ancillary qubits and input state are coupled with two CNOT gates. After the unitary evolution, both the ancillary qubits 3 and 4 are measured along the $\{|0\rangle, |1\rangle\}$ basis, whereas qubits 1 and 2 are then projected to a disturbed version of the original maximal entangled state ρ'.

For simplicity, let $\mathbf{C_{i,j}}$ denote the CNOT gate with the index $i(j)$ indicating the control (target) qubit. $\mathbf{H_i}$ represents the Hadamard gate acting on the qubit i. The unitary evolution of the quantum circuit in figure 1 follows

$$\mathcal{U} = \mathbf{C_{21}H_2C_{13}C_{24}H_2C_{21}}. \tag{29}$$

Assuming the measurement outcomes of qubits 3 and 4 are (m, n), the measurement operators A_{mn} can be given by

$$A_{m,n} = (I_{12} \otimes {}_{34}\langle m, n|)\,\mathcal{U}\,(I_{12} \otimes |w\rangle_{34})\,. \tag{30}$$

Particularly, in $D = 2$ cases, the measurement outcome (m, n) will only take values from $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. In the standard basis of $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, we can express the corresponding Kraus operators as follows:

$$
\begin{aligned}
A_{00} &= \frac{1}{2}\begin{pmatrix} \cos\theta + \gamma\sin\theta & & & \cos\theta \\ & \gamma\sin\theta & & \\ & & \gamma\sin\theta & \\ \cos\theta & & & \cos\theta + \gamma\sin\theta \end{pmatrix}, \\
A_{01} &= \frac{1}{2}\begin{pmatrix} \cos\theta + \gamma\sin\theta & & & -\cos\theta \\ & \gamma\sin\theta & & \\ & & \gamma\sin\theta & \\ -\cos\theta & & & \cos\theta + \gamma\sin\theta \end{pmatrix}, \\
A_{10} &= \frac{1}{2}\begin{pmatrix} \gamma\sin\theta & & & \\ & \cos\theta + \gamma\sin\theta & \cos\theta & \\ & \cos\theta & \cos\theta + \gamma\sin\theta & \\ & & & \gamma\sin\theta \end{pmatrix}, \\
A_{11} &= \frac{1}{2}\begin{pmatrix} \gamma\sin\theta & & & \\ & \cos\theta + \gamma\sin\theta & -\cos\theta & \\ & -\cos\theta & \cos\theta + \gamma\sin\theta & \\ & & & \gamma\sin\theta \end{pmatrix}.
\end{aligned}
\tag{31}
$$

It should be noted that these operators coincide well with those obtained in [7]. Plugging equation (31) into equation (11), one can find that the output fidelity for operators $\{A_{00}, A_{01}, A_{10}, A_{11}\}$ will be

$$
\mathcal{F} = 1 - \frac{\cos^2\theta}{2}. \tag{32}
$$

In order to maximize the estimation fidelity, the optimal inference rule can be constructed as follows. When the measurement outcome (m, n) is observed, one infers that the state ϕ_r is A_{mn}'s *dominant eigenstate* (the eigenstate corresponding to the maximal eigenvalue):

$$
\begin{aligned}
(0,0) &\longrightarrow |\phi_{00}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
(0,1) &\longrightarrow |\phi_{01}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
(1,0) &\longrightarrow |\phi_{10}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
(1,1) &\longrightarrow |\phi_{11}\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned}
\tag{33}
$$

It is straightforward to verify, from equations (18) and (31) , that the estimation fidelity

$$
\mathcal{G} = \frac{1}{2} - \frac{\gamma^2\sin^2\theta}{4}. \tag{34}
$$

Equations (32) and (34) say that $\mathcal{F}$ and $\mathcal{G}$ are both determined by the parameter θ and any ratio between them can be achieved by suitable superposition of the ancillary qubits. First, let us consider two extreme cases. (1) $\theta = \pi/2$. At this point, the original state is completely preserved and thus $\mathcal{F} = 1$. However, at the same time, the inference rule has to be a randomly guessing and estimation fidelity is $\mathcal{G} = 1/4$. This corresponds to the case of the non-informative measurement. The other extreme case is (2) $\theta = 0$, which gives $\mathcal{F} = \mathcal{G} = 1/2$. The information of the unknown entangled state is maximally extracted whereas the output state is also maximally disturbed. In this case, the optimal measurement is the projective entanglement measurement or Bell measurement. $0 < \theta < \pi/2$ is the intermediate case and the optimal tradeoff interpolates smoothly between those two extreme cases.

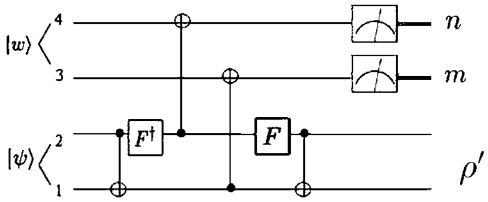

Figure 2. The quantum circuit for achieving the optimal tradeoff for an arbitrary D-dimensional maximally entangled state.

Upon eliminating θ, we obtain

$$\mathcal{F} + 2\mathcal{G} - \sqrt{(1-2\mathcal{G})(1-\mathcal{F})} = \tfrac{3}{2}. \tag{35}$$

This $\mathcal{F}-\mathcal{G}$ relation in equation (35) corresponds to the bound equation (28) with the equal sign and thus confirms the statement that the scheme in figure 1 is actually optimal.

We will now explore the possibility of generalizing the optimal measurement scheme shown above to an arbitrary D-dimensional situations. But first of all, we need to introduce some elementary quantum logic gates for general operations. The first one is the two-qudit CNOT gate [27]. This is the one of the fundamental bipartite interaction. In the computation basis $\{|m\rangle | m = 0, 1, \ldots, D-1\}$, such a unitary interaction is given by

$$\mathbf{C_{i,j}} = \sum_{i,j} |m, m \oplus n\rangle\langle m, n|, \tag{36}$$

where $\oplus$ denotes sum *modulo* D. This is a straightforward generalization of CNOT gate from its 2-dimensional cases. Another operation that should be notified is the Fourier transformation. This is a single qudit unitary operation

$$\mathbf{F} = \frac{1}{\sqrt{D}} \sum_{m,n=0}^{D-1} \mathrm{e}^{\frac{\mathrm{i}2\pi mn}{D}} |m\rangle\langle n|, \tag{37}$$

with its inverse given by

$$\mathbf{F^{-1}} = \mathbf{F}^{\dagger} = \frac{1}{\sqrt{D}} \sum_{m,n} \mathrm{e}^{-\frac{\mathrm{i}2\pi mn}{D}} |m\rangle\langle n|. \tag{38}$$

The D-dimensional measurement scheme shown in figure 2 is similar to that of figure 1 except that the Hadamard gate is replaced by the Fourier gate and CNOT operation is automatically advanced to its D-dimensional variant. Similarly, auxiliary qudits 3 and 4 are initially prepared in the partial entanglement

$$|w\rangle_{34} = \cos\theta|00\rangle + \gamma\sin\theta|++\rangle, \qquad \gamma = \frac{\sqrt{1 + D^2\tan^2\theta} - 1}{D\tan\theta},$$

where $|+\rangle = \frac{1}{\sqrt{D}}\sum_{i=0}^{D-1}|\mathrm{i}\rangle$, and θ takes value from $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

The corresponding measurement operators in figure 2 can be obtained from equation (30) with $\mathcal{U} = \mathbf{C_{21}}.\mathbf{F_2}.\mathbf{C_{13}}.\mathbf{C_{24}}.\mathbf{F_2}.\mathbf{C_{21}}$. After direct calculation, we have

$$A_{mn} = \frac{1}{D}\left(\cos\theta|U_{mn}\rangle\rangle\langle\langle U_{mn}| + \gamma\sin\theta\cdot I\otimes I\right), \tag{39}$$

in which we define

$$U_{mn} = \sum_{k=0}^{D-1} \mathrm{e}^{\frac{\mathrm{i}2\pi nk}{D}} |k\rangle\langle k \oplus m|, \tag{40}$$

with $m, n = 0, 1, \ldots, D-1$ denoting the measurement results of qudits 3 and 4, respectively. This implements the Kraus operators which interpolate between the identity map and projective measurement for the maximal entanglement state, and which was originally proven to be optimal in estimating the maximal entanglement state in [7].

Assuming the inference rule to be

$$(m, n) \rightarrow |\phi_{mn}\rangle \equiv \frac{1}{\sqrt{D}}|U_{mn}\rangle\rangle = \frac{1}{\sqrt{D}}\sum_{k} \mathrm{e}^{\frac{\mathrm{i}2\pi nk}{D}}|k\rangle|k \oplus m\rangle, \tag{41}$$

one can evaluate the operation fidelity and estimation fidelity from equations (11), (18) and (39) that

$$\mathcal{F} = 1 - \frac{D^2-2}{D^2}\cos^2\theta, \qquad \mathcal{G} = \frac{2-\gamma^2\sin^2\theta}{D^2}. \tag{42}$$

It can easily be checked that the equality sign in equation (28) is saturated and the scheme presented in figure 2 is the optimal quantum circuit.

4. Conclusion

In conclusion, we have retrieved information–disturbance tradeoff for estimating an unknown maximally entangled state. Moreover, we also show the quantum scheme for physically achieving such tradeoff. The scheme is based on a quantum circuit that involves single- and two-particle logic gates and ancillary particles and single-particle measurements.

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